The Chow-Lam Form

joint work with Bernd Sturmfels

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Slides: https://lizziepratt.com/notes

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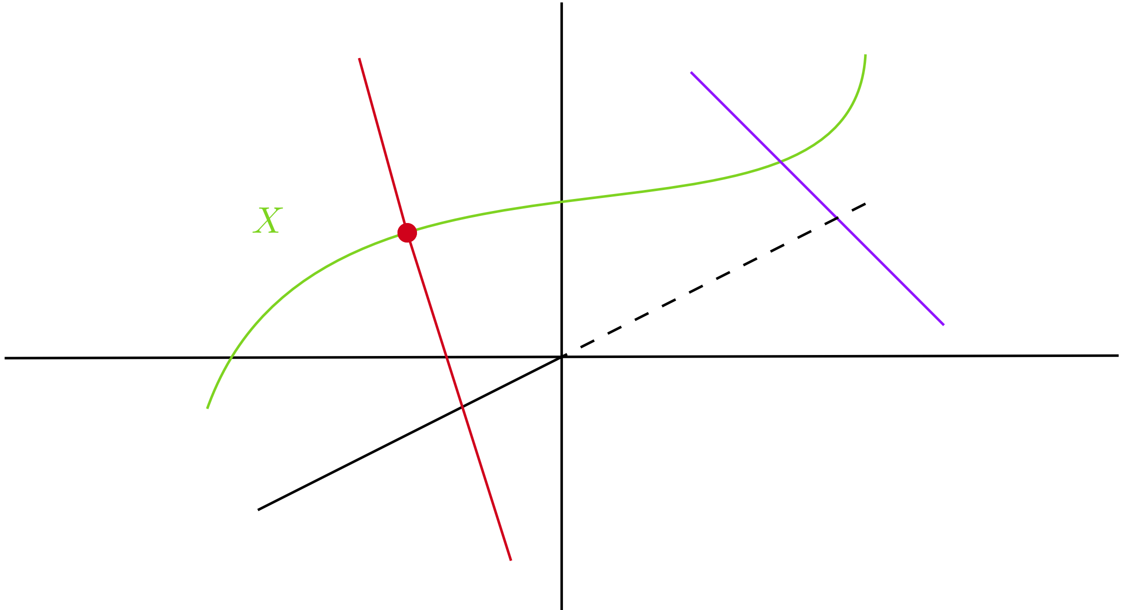
The Chow Form

Definition (Chow form)

Let $X \subset \mathbb{P}^{n-1}$ be a d-dimensional projective variety. The *Chow locus* of X is

$$\mathcal{C}_X = \{L \in \mathrm{Gr}(n-d-1, n) : X \cap L \neq \varnothing\}.$$

The *Chow form* C_X is the defining equation of $\mathcal{C}_X$.



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Example (Hypersurface)

The Chow form of a hypersurface $V(F)$ is F.

Example (Linear space)

The Chow locus of a linear space is a Schubert divisor.

Coordinate Systems

A linear space L can represented multiple ways.

- Primal : as the kernel of an $(n-k)\times n$ matrix
- Dual : as the rowspan of a $k \times n$ matrix

The primal and dual Plücker coordinates are the maximal minors of these matrices.

Example (Coordinates on Gr$(3,5)$)

$$\begin{array}{cccccccccc} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45} \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123} \end{array}$$

The twisted cubic

The Chow locus is

$$\mathcal{C}_X = \{L \in \mathrm{Gr}(2,4) \text{ such that the line } L \text{ meets } X \text{ in } \mathbb{P}^3\}.$$

The Chow form is the determinant of the *Bézout matrix* :

$$C_X = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}$$

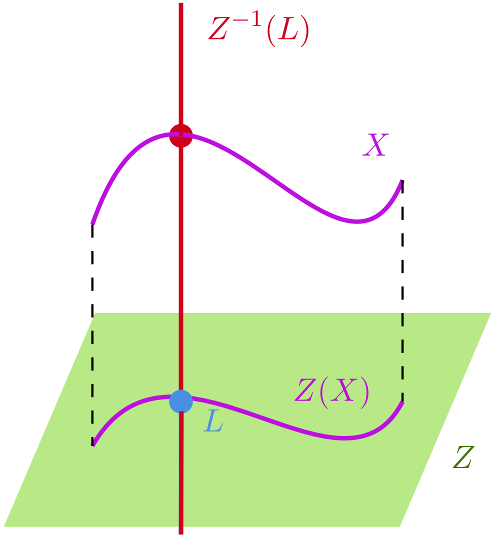
For more on curves and resultants, see [David Eisenbud and Frank-Olaf Schreyer: *Resultants and Chow Forms via Exterior Syzygies*, 2003]

Theorem: Projection

Let L be a linear subspace of $\mathbb{P}^{n-1}$. Let Z be a $r \times n$ matrix, where $r \leq \dim X - 2$. We get a projection

$$Z : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}.$$

Then $L \in \mathcal{C}_{Z(X)} \iff Z^{-1}(L) \in \mathcal{C}_X$.



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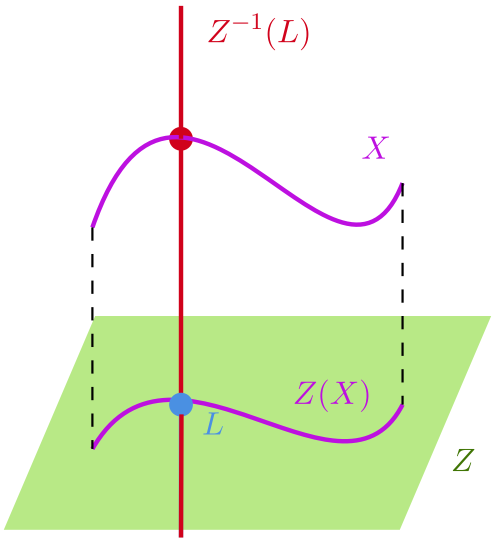


Q: How do we compute Plücker coordinates of the fiber $Z^{-1}(L)$ in terms of Plücker coordinates of L and matrix entries of Z?

Twistor coordinates

Proposition (Twistor Coordinates)

Let L be given in dual coordinates, as the column span of an $r \times (r-d-1)$ matrix. The primal Plucker coordinate $p_I(Z^{-1}(L))$ for $|I| = d+1$ is the Ith twistor coordinate of Z and L, given by

$$[Z : L]_I := \det[Z_I \ L].$$

Example

Let $n = 4, r = 3$, and $d = 1$. Then

$$p_{12}(Z^{-1}(L)) = \det[Z_1 \ Z_2 \ L],$$
$$p_{13}(Z^{-1}(L)) = \det[Z_1 \ Z_3 \ L],$$
$$\vdots$$

The twisted cubic revisited

Recall the Chow form:

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}$$

We make the substitution where the p_{ij} index 3×3 minors of

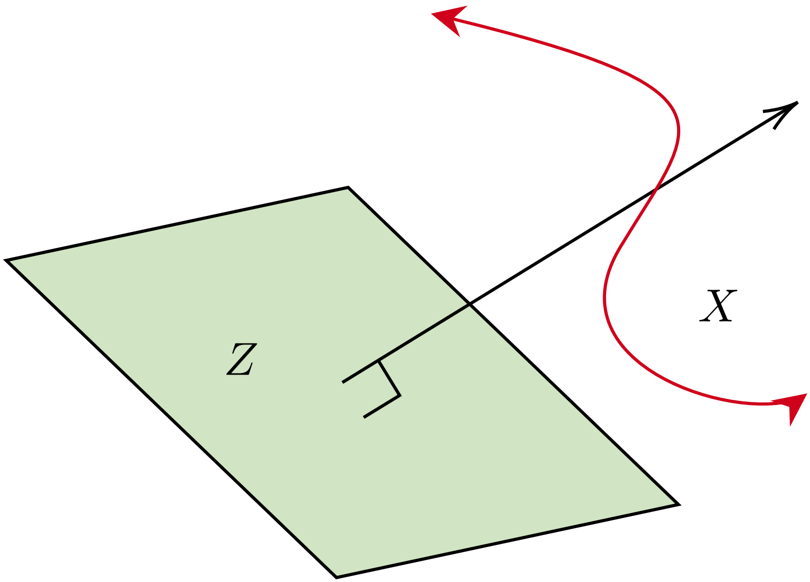
$$[Z|L] = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & l_1 \\ z_{21} & z_{22} & z_{23} & z_{24} & l_2 \\ z_{31} & z_{32} & z_{33} & z_{34} & l_3 \end{bmatrix}$$

involving l. Then $p_{ij}(Z^{-1}(L)) = [Z|L]_{ij}$

$$= (z_{1i}z_{2j} - z_{2i}z_{1j})l_3 - (z_{1i}z_{3j} - z_{3i}z_{1j})l_2 + (z_{2i}z_{3j} - z_{3i}z_{2j})l_1.$$

Now $C_{Z(X)}(L) = C_X(Z^{-1}(L))$ is a function of $[l_1 : l_2 : l_3]$ on $\mathbb{P}^3$ and the matrix entries of Z.

The twisted cubic revisited



The Chow form $C_{Z(X)}(L)$ computes universal projection equations.

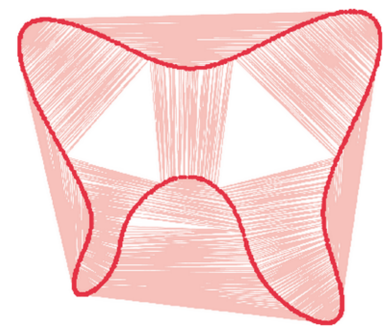
A new context: Grassmannians

Inspired by [Thomas Lam: *Totally nonnegative Grassmannian and Grassmann polytopes*, 2015]

Let Z be a $r \times n$ matrix. This gives a map

$$Z : \mathrm{Gr}(k,n) \dashrightarrow \mathrm{Gr}(k,r)$$
$$[M] \mapsto [ZM]$$

Can we compute "universal" equations of projected varieties which are codimension 1 in the target?



The Chow-Lam form

Definition (Grassmannian)

For a linear space P in $\mathbb{P}^{n-1}$, let $\mathrm{Gr}(k, P)$ denote $(k-1)$-planes in $\mathbb{P}^{n-1}$ contained in P.

Definition (Chow-Lam form)

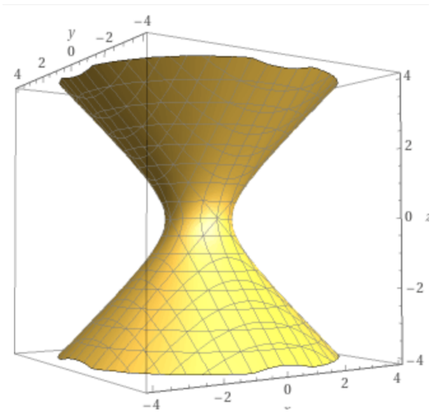
Let $\mathcal{V}$ be a variety of dimension $k(r-k)-1$ for some $r \leq n$. We define the *Chow-Lam locus* to be

$$\mathcal{CL}_{\mathcal{V}} := \{P \in \mathrm{Gr}(k+n-r, n) : \mathrm{Gr}(k, P) \cap \mathcal{V} \neq \varnothing\},$$

i.e. "spaces P which have a subspace Q, where Q is a point of $\mathcal{V}$." When $\mathcal{CL}_{\mathcal{V}}$ has codimension 1, its defining equation is the *Chow-Lam form* $CL_{\mathcal{V}}$.

Example: Ruled surface

Let $\mathcal{V} \subset \text{Gr}(2,4)$ be a curve, and let S be the surface swept out by lines parameterized by $\mathcal{V}$.



- Here $n = 4, k = 2, r = 3$, so $\mathcal{CL}_{\mathcal{V}}$ is a surface in $\mathrm{Gr}(4-3+2,4) = \mathrm{Gr}(3,4) = (\mathbb{P}^3)^{\vee}$.

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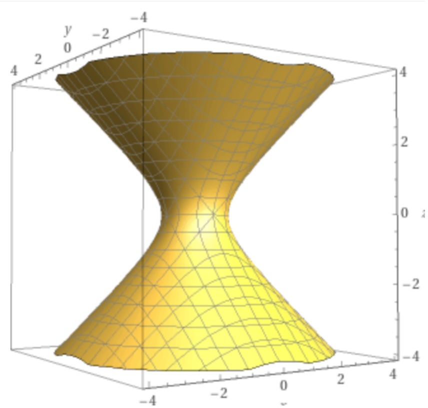

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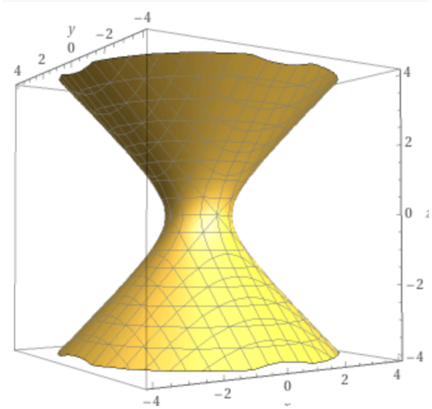

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- It parametrizes planes P in $\mathbb{P}^3$ which contain a line Q from $\mathcal{V}$.
- Turns out $\mathcal{CL}_{\mathcal{V}}$ is the projective dual of S, i.e. the variety of tangent planes.

Projection Formula

Proposition (P-Sturmfels)

The Chow-Lam form $\mathrm{CL}_{Z(\mathcal{V})}$ *is obtained from the Chow-Lam form* $\mathrm{CL}_{\mathcal{V}}$ *by replacing the primal Plücker coordinates with twistor coordinates:*

$$p_I \;=\; \det[Z_I \; L] \quad \text{for } I \subset \binom{n}{r-k}.$$

This expresses $\mathrm{CL}_{Z(\mathcal{V})}$ *in dual Plücker coordinates on* $\mathrm{Gr}(k,r)$*, given by the* $r \times k$ *matrix* L*.*

Example: Positroid varieties

Fix $k{=}2, n{=}9, r{=}7$ and the positroid variety

$$\mathcal{V} \;=\; V(q_{12}, q_{13}, q_{23}, q_{45}, q_{67}, q_{89}) \;\subset\; \mathrm{Gr}(2,9).$$

This variety is dimension 9 and consists of rowspans of 2×9 matrices $L = [L_1 ... L_9]$ with $\mathrm{rk}(L_{123}) = \mathrm{rk}(L_{45}) = \mathrm{rk}(L_{67}) = \mathrm{rk}(L_{89}) = 1$. Then $\mathcal{CL}_{\mathcal{V}}$ is in $\mathrm{Gr}(4,9)$.

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Let $X = [X_1 ... X_9]$ be a 4×9 matrix. Note that $L \subset X$ if and only if there exists a 2×4 matrix T with $L = TX$. Then TX is in $\mathcal{V}$ iff

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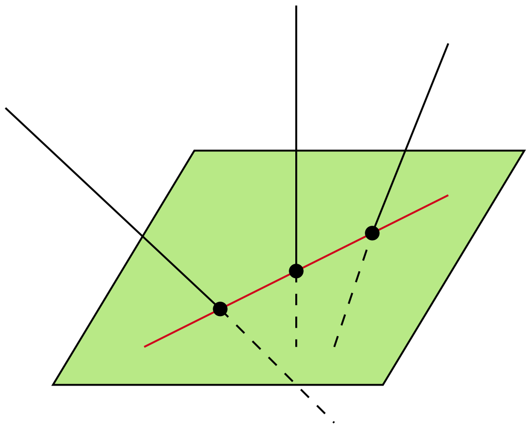
Geometrically: The line in $\mathbb{P}^3$ given by T is contained in the plane X_{123} and intersects the lines X_{45}, X_{67}, and X_{89}.

Example: Positroid varieties

For which X does there exists a line T in $\mathbb{P}^3$ contained in the plane X_{123} and intersecting the lines X_{45}, X_{67}, and X_{89}?



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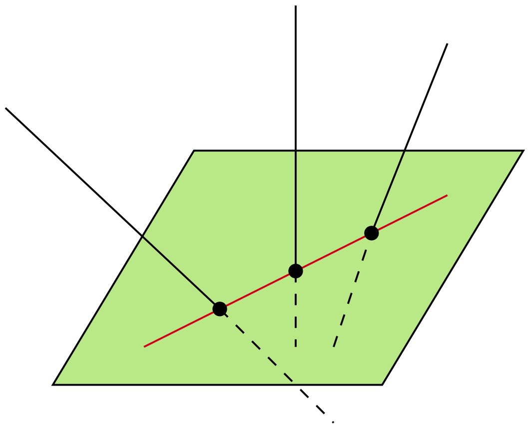


Answer : When the points $X_{123} \cap X_{45}, X_{123} \cap X_{67}, X_{123} \cap X_{89}$ are collinear! In Plücker coordinates, we have

$$X_{123} \cap X_{ij} = q_{23ij} X_1 - q_{13ij} X_2 + q_{12ij} X_3.$$

The Chow-Lam form is

$$\mathrm{CL}_{\mathcal{V}} = \det \begin{bmatrix} q_{2345} & -q_{1345} & q_{1245} \\ q_{2367} & -q_{1367} & q_{1267} \\ q_{2389} & -q_{1389} & q_{1289} \end{bmatrix}.$$

Example: Positroid varieties

Fix $k = 2, n = 10, r = 8$ and consider the positroid variety

$$\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}, q_{90}) \subset \mathsf{Gr}(2, 10).$$

By similar reasoning, the Chow-Lam locus is given by matrices $X = \begin{bmatrix} X_1 & \dots & X_9 & X_0 \end{bmatrix}$ where the five lines $X_{12}, X_{34}, X_{56}, X_{78}, X_{90}$ have a common transversal. This codimension 1 condition is given by the Chow-Lam form

$$\mathrm{CL}_{\mathcal{V}} = \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} & q_{1290} \\ q_{1234} & 0 & q_{3456} & q_{3478} & q_{3490} \\ q_{1256} & q_{3456} & 0 & q_{5678} & q_{5690} \\ q_{1278} & q_{3478} & q_{5678} & 0 & q_{7890} \\ q_{1290} & q_{3490} & q_{5690} & q_{7890} & 0 \end{bmatrix}.$$

Hurwitz forms and higher Chow forms

Higher Chow forms characterize linear spaces that intersect $\mathcal{V}$ non-transversally. These are also known as coisotropic hypersurfaces.

Definition (Hurwitz locus)

The *Hurwitz locus* of a projective variety X of dimension d consists of linear spaces of codimension d which intersect X non-transversally.

Definition (Hurwitz-Lam locus)

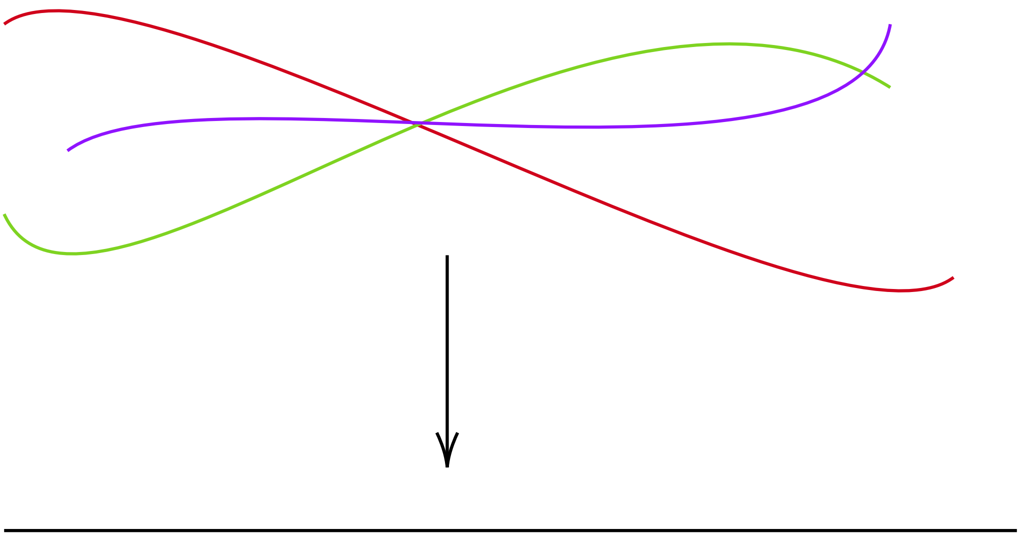
Let $\mathcal{V}$ have dimension $k(r-k)$ for some r. The *Hurwitz-Lam locus* of $\mathcal{V}$ is

$$\mathcal{HL}_{\mathcal{V}} \;=\; \{\, P \in \mathrm{Gr}(k+n-r,n) \,:\, \mathcal{V} \cap \mathsf{Gr}(k,P) \text{ is not transverse} \}.$$

Hurwitz-Lam forms

Theorem (P-Sturmfels, Computing Branch Loci)

The branch locus of Z : Gr$(k,n) \dashrightarrow$ Gr(k,r) in $\mathcal{V}$ is a hypersurface in $\mathrm{Gr}(k,r)$, and its equation is obtained from the Hurwitz-Lam form by replacing the primal Plücker coordinates with twistor coordinates.



Example: Four mass box

Consider the positroid variety given by

$$\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}) \subset \mathsf{Gr}(2, 8).$$

Its Hurwitz-Lam form is in $\mathrm{Gr}(4, 8)$ and consists of matrices $X = [X_1 ... X_8]$ where the system of equations

$$\mathsf{rk}(TX_{12}) = \mathsf{rk}(TX_{34}) = \mathsf{rk}(TX_{56}) = \mathsf{rk}(TX_{78}) = 1$$

has fewer solutions than expected.

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Geometrically: Matrices X such that the lines $X_{12}, X_{34}, X_{56}, X_{78}$ have a single common transversal in $\mathbb{P}^3$.

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Geometrically: Matrices X such that the lines $X_{12}, X_{34}, X_{56}, X_{78}$ have a single common transversal in $\mathbb{P}^3$. This condition is given by

$$\mathrm{HL}_{\mathcal{V}} = \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} \\ q_{1234} & 0 & q_{3456} & q_{3478} \\ q_{1256} & q_{3456} & 0 & q_{5678} \\ q_{1278} & q_{3478} & q_{5678} & 0 \end{bmatrix}.$$

Thank you for listening!

Example: Positroid variety

Proposition (P-Sturmfels)

The Hurwitz-Lam form of the positroid variety $\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}) \subset \mathrm{Gr}(2,8)$ *equals*

$$\begin{aligned} \mathrm{HL}_{\mathcal{V}} \;=\; & p_{1235}^2 p_{4678}^2 + p_{1236}^2 p_{4578}^2 + p_{1245}^2 p_{3678}^2 + p_{1246}^2 p_{3578}^2 \\ & + 4 p_{1235} p_{1246} p_{3578} p_{4678} - 2\,(p_{1234} p_{1256} p_{3478} p_{5678} + p_{1234} p_{1256} p_{3578} p_{4678} \\ & + p_{1235} p_{1236} p_{4578} p_{4678} + p_{1235} p_{1245} p_{3678} p_{4678} + p_{1235} p_{1246} p_{3478} p_{5678} \\ & + p_{1236} p_{1246} p_{3578} p_{4578} + p_{1245} p_{1246} p_{3578} p_{3678}). \end{aligned}$$

Geometrically: The Grasstope $Z(\mathcal{V}) \subset$ Gr$(2,6)$ has a degree four and codimension two boundary given by this ramification locus, which is NOT itself the projection of a positroid variety.

Example: Schubert varieties

Consider the following Schubert varieties in Gr$(2,5)$ of dimension $3 = 2(4-2)-1$:

- $\Sigma_{24}^\circ = \left\{ \text{rowspan} \begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix} \right\}$
- $\Sigma_{15}^\circ = \left\{ \text{rowspan} \begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$

The Chow locus lives in Gr$(5+2-4,5) =$ Gr$(3,5)$. It turns out that

- $\mathcal{CL}_{\Sigma_{24}} = V(p_{45})$
- $\mathcal{CL}_{\Sigma_{15}} = V(p_{15}, p_{25}, p_{35}, p_{45})$

Thus we get an example of a Chow-Lam locus with codimension higher than 1.

Example: Schubert varieties

Connection with projection.

- $\Sigma_{24}^{\circ} = \left\{ \text{rowspan} \begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix} \right\}$
- $\Sigma_{15}^{\circ} = \left\{ \text{rowspan} \begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$

Fix a 4×5 matrix Z. Then points in $Z(\Sigma_{24})$ and $Z(\Sigma_{15})$ look like

- rowspan $\begin{bmatrix} Z_2 + \star Z_3 + \star Z_5 \\ Z_4 + \star Z_5 \end{bmatrix}$
- rowspan $\begin{bmatrix} Z_1 + \star Z_2 + \star Z_3 + \star Z_4 \\ Z_5 \end{bmatrix}$

These are "lines which intersect the line span(Z_4, Z_5) in $\mathbb{P}^3$" and "lines which pass through the point Z_5 in $\mathbb{P}^3$". In terms of a line rowspan(L), their equations are respectively

- $\det[Z_4 Z_5 | L] = 0$
- $\det[Z_i Z_5 | L] = 0$ for $i = 1, 2, 3, 4$.